# The Chemistry of Platonic Triangles: Problems in the Interpretation of the *Timaeus*

*D. Robert Lloyd*

**Abstract:** Plato's geometrical theory of what we now call chemistry, set out in the *Timaeus*, uses triangles, his *stoicheia*, as the fundamental units with which he constructs his four elements. A paper claiming that these triangles can be divided indefinitely is criticized; the claim of an error here in the commentary by F.M. Cornford is unfounded. Plato's constructions of the elements are analyzed using simple point group theory. His procedure generates fully symmetric polyhedra, but Cornford's 'simpler' alternatives generate polyhedra with low symmetries and multiple isomeric forms. However, Cornford's principle of constructing larger triangles by assembling smaller ones is still valid.

**Keywords:** *Plato's chemistry, elements, F. M. Cornford, group theory, Platonic solids*.

## Introduction

The *Timaeus*[1] has been the object of study and commentary ever since its appearance,[2] and was long regarded as the most significant work of Plato. As an indication of this importance through the late middle ages and beyond, Raphael (1483-1520), in his fresco 'The School of Athens', characterizes Plato by showing him holding this work in his hand (apparently written in Italian!). In his extraordinary work, Plato sets out to provide a "reasonable account" (*eikos logos*) of the whole of Creation by the 'Demiurge', both at the cosmic scale and also at the microscopic level. It is important to appreciate, given the intervening Christian centuries, that neither is this is a *creatio ex nihilo,* nor is the Demiurge omnipotent. Some versions in English give him the literally translated name of 'The Craftsman'; he works with the pre-existing materials available to him, and I will return to this point.

In the context of this Journal, Plato's account of the nature of the four elements is his most significant contribution. In the second part of the *Timaeus*, Plato sets out a scheme for what we now call chemistry, biochemistry,

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and human biology. Concerning this part, the view of Böhme and Böhme (1996, p. 100) is that *Timaeus*

> ist philosophie- und wissenschaftsgeschichtlich von außerordentlicher Bedeutung, weil er fast der einzige Text ist, der von den Dialogen Platons durchgängig in Europa den Gelehrten bekannt war. Dann aber insbesondere, weil er eine der Quellen war, aus denen sich zur Zeit des alexandrinischen Neuplatonismus die Alchemie entwickelte. Als solchen kamen die antiken Lehren von der Materie, vermittelt über die Araber, ins lateinische Mittelalter. Auf dem Hintergrund der Alchemie entwickelte sich vom 17. Jahrhundert an die neuzeitliche Chemie.

Thus, in their opinion, the dialogue counts as one of the foundations of our modern chemistry, *via* its influence on early alchemy. An understanding of chemistry as Plato saw it is thus of some significance. The general outlines are clear. His theory is partly based on the earlier four elements,[3] but there are two very important additions to previous theories. The first of these is that each of the traditional elements is to be associated with the form of one of the regular polyhedra, which we still call the Platonic Solids. He assigns the dodecahedron to the entire cosmos, and the other four as elements. These four are constructed using his truly elementary units, the *stoicheia*, two particular right triangles. Chemistry, as he understands it, is based on the most advanced mathematics available to him, geometry. There are obvious parallels with our modern situation, in which quantum mechanics, at some level or other, is fundamental to our understanding of the subject.

The second addition, particularly relevant to the first part of this paper, is that each polyhedral form is the prototype for a whole range of elements, where differently sized units have notably different properties. For example the general class of elements which he calls "fire" is associated with the regular tetrahedron. Depending on the size of the tetrahedron, this can be conventional fire, visible light (Tim. 58C) of various colors (67D), or it can be a sort of radiation believed to be emitted from the eye as part of the process of vision (45B-C). Similarly ordinary water and gold are both considered "waters", associated with icosahedra: various forms of "air" are all to be associated with different sizes of octahedra. (For a summary of the differently sized elements, see Brisson and Meyerstein 1995, pp. 53-4, and Brisson 2003, pp. 15-16.)

These three different polyhedra are considered to be formed by assembling equilateral triangles. These triangles in turn are made up by assembling six of what Plato considers truly elemental units, *stoicheia*, scalene triangles with angles of 30°, 60°, and 90°. (The construction is illustrated in Figure 1b). It is very important for Plato that because of these common triangular units, his elements can interconvert (54C-D), unlike those of the atomic theories of the time. He has problems with the fourth element type, earth, because he has to assign this to the cube, which, being made up of right-angled isosceles triangles forming squares, cannot interconvert with the other three. Thus, within the *Timaeus* theory, there are two shapes of elemental triangles, and each of these exists in more than one size.

A particularly significant commentary on the text of the *Timaeus* is F.M. Cornford's *Plato's Cosmology* (1937), which, despite its age, is still probably the pre-eminent work in the English language.[4] This paper is concerned with two particular aspects of Cornford's analysis of the way in which Plato constructs the three-dimensional regular units which make up the elements. Cornford did not accept that Plato's specified geometrical construction of the equilateral triangle from six *stoicheia* was unique; rather it was to be understood as a 'sample' of many different constructions using varying numbers of *stoicheia*. In the second half of this paper I show that Plato's specified construction has to be taken more seriously; it is in fact the only way to construct three-dimensional units with the true symmetry of the regular bodies. Nevertheless, Cornford's 'building block' approach, if used with a proper consideration of symmetry, is still consistent with Plato's specification, even though some of his proposed structures have low symmetry, and therefore have to be discarded.

In the earlier part of this paper, I deal with a related problem. Cornford's proposal is that Plato's differently sized equilateral triangles should be considered as all made up by assembling different numbers of the same size of elementary scalene triangle. This has had wide acceptance, at least among writers in English.[5] However, an earlier paper in this Journal, 'A Potential Infinity of Triangle Types: On the Chemistry of Plato's *Timaeus*', by J. Visintainer, makes some remarkable claims concerning this point in Cornford's commentary. The abstract of this work begins (Visintainer 1998, p. 117):

> Francis Cornford's assertion that there must be a smallest elemental triangle in the Platonic chemistry of the *Timaeus* is overturned in this paper.

Visintainer's proposal implies that Cornford's scheme of building up larger triangles from the same basic unit is unnecessary – instead, differently sized scalene triangles can be generated by subdivision of larger ones. Thus there need be no smallest elemental triangle; any particular one can be indefinitely subdivided, giving his "potential infinity" of triangles. On a variety of grounds, I contend that this is not the case, and these arguments form the subject of the first half of my paper.

# Part I. Finite or Infinitesimal Triangles?

## 1. Triangles in the commentary by Cornford

The quotation above implies that Cornford made a specific new proposal "that there must be a smallest elemental triangle". Reading his text, it seems likely that he would have been surprised by the accusation. In fact he is discussing the construction of *larger* triangles, and his geometrical proof that there is an upward-going series of these is most easily shown with the device of a downward-going argument of subdividing triangles. He may have been a little careless in speaking of 'Plato' not continuing a process of division, since there is no mention of any such division in the *Timaeus* text here; this division is Cornford's own.[6] However, a similar criticism can be leveled at the statement (Visintainer 1998, p. 123) "Plato never explicitly states that the process of division of equilateral triangles does not continue indefinitely." Since Plato has not discussed such division, there is no obvious reason why he should.

## 2. Infinitesimals?

It is certainly clear that Cornford believes the elementary triangles, Plato's *stoicheia*, to be finite, and not capable of being divided, but this was hardly a new proposal of his. In the commentary by Taylor (1928, pp. 372-3), published a decade earlier, there is a much more definite statement of the point. "It is assumed [by *Timaeus*] that the faces of the molecules can only be physically broken up along the lines of cleavage which are the sides of 'elementary' triangles. The two 'elementary' triangles are physically the ultimates of the theory, and therefore indivisibles, though they have finite areas, are not 'infinitesimals'." If there is to be a discussion of divisibility, and infinitesimals, potential or otherwise, it is this text rather than Cornford's which should be considered.

Going further back, there is evidence from Aristotle, who had been Plato's student, that Plato intended the triangles to be indivisible. According to Vlastos (1975, p. 67), "Aristotle takes it for granted that Plato's theory is a variant of the atomic hypothesis of Leucippus and Democritus", commenting that "both [theories] uphold 'indivisible magnitudes'". There is a discussion of this point in the *Timaeus* by Sorabji (1983), who states that "Plato distinguishes between perceptible, mathematical and ideal entities", and that although he does accept the possibility of infinite divisibility, this is only for the latter entities, not for real, "perceptible" ones. Concerning the triangles, the "elementary particles are all rectilinear and, we may add, do not fall below a certain size." More recently Gregory (2000, p. 203) has stated that "The evidence of Aristotle then is very much in favor of the idea that the stoicheic triangles do not undergo any sort of *alloiôsis*". Neither of the comments by Vlastos and Sorabji seems to have been taken into account by Visintainer; the work proceeds to a demonstration that Plato could have used a scheme of division leading to infinitesimal (or "potentially infinitesimal") triangles in his scheme, but serious questions remain. Firstly, bearing in mind the remarks above, would Plato have had any interest in using such a procedure for a description of the real world? Secondly, did he have any need for it? I turn to this question now.

## 3. Triangles and mixtures, elements and compounds

The work includes the statement "I will examine why [...] the triangles are not given any definite size by Plato" (Visintainer 1998, p. 117). Perhaps the most obvious reason is that Plato did not need to, since so long as the largest unit is below the level of visibility, any finite scale will do; indeed it is not even clear how he could have made such an assignment of microscopic sizes. However the main motivation for this work appears to be indicated by the multiple references in the paper to the supposed 'call' of Plato for an 'endless diversity' of elemental triangles. The basis for this claim is the following footnote (Visintainer 1998, p. 128, note 4):

> The Greek at 57d reads "*tên poikilian estin apeira*". '*poikilian*' refers there to the many different or motley units from which the solids are ultimately made, *i.e.* the triangles, be they of the right scalene type or the half square type. These units are *(estin)* without limit *(apeira)*.

The translations of the words (from 57D 4-5) are correct, but the deduction about triangles is not. An important preceding clause has been omitted, and '*poikilian*' does *not* refer to triangles. Cornford's translation of this part of the text has "Hence, when they are mixed with themselves or with one another, there is an endless diversity". Here 'they' refers to triangles and to the "uncompounded and primary bodies" (*akrata kai prôta sômata,* see the previous sentence) which they make up, but the variety is of *mixtures*, not of triangles or bodies. The "endless diversity" refers not to the triangles at all, but to the complex mixtures possible with the many different elements. In addition to the translation, on p. 230, Cornford also makes this point clearly and quite specifically on p. 231, and again in his notes introducing the next section (p. 246). On this interpretation[7] there is clearly *no* requirement for an 'endless diversity' or 'infinite variety' of triangles. Not only has Plato not "called for" this; he has not even suggested it.

In modern terms, it is perfectly possible to state that our ninety or so reactive elements can give rise to an "endless diversity" of compounds. Indeed, given the non-stoicheiometric nature of the phases, this is true just for iron and oxygen, and many other pairs. Plato has not specified anything about the stoicheiometry of his 'mixtures', and in the phrase "*tên poikilian estin apeira*", he is saying no more than that there is no limit to the number of possible combinations of his elements.

Although the quoted phrase carries no information about the triangles/ primary bodies, the previous rather long sentence does. We are told that there are "numerically as many as there are varieties within a given form".[8] Presumably Plato does not know how many species there might be, but there is no suggestion that these numbers might be infinite; the examples he gives suggest that, far from there being an 'endless diversity', the actual numbers of elements, and of corresponding triangles, would probably be rather small (Brisson and Meyerstein 1995, pp. 53-4).

## 4. *Can* the atom be split?

The proposal (see Figures 9 and 10 of Visintainer 1998, and the related text) is that if a structure such as an equilateral triangle exists, it can be sub-divided indefinitely, taking no account of the way in which Plato specified that it be constructed from *elementary* scalene triangles, even though Plato considers the function of these triangles to be similar to that of letters making up syllables (47C). Indeed, Plato's use of the word *stoicheion*, originally 'letter', makes it immediately apparent that there is a serious problem here. At 47B-C, he has rejected earlier atomic schemes precisely because they have not come up with a truly fundamental unit, a 'letter' of the alphabet, while he has (Zeyl 2000, p. 37, notes 50, 51). Using Plato's metaphor, it is doubtful whether any significance could be attached to a fraction of a letter, and certainly none could apply if it were to be divided indefinitely.

We also need to consider the Demiurge and his limitations. The triangles are a part of his working material. Using them, he forms them into the elements which are then as nearly perfect as possible (53B). The source of the triangles is not specified, but whether they are a part of the initial chaotic 'stuff' of the cosmos before he sets it in order, or he is making them as a part of creation following the perfect model (28A-B), some of them are of higher quality than others. Accordingly he has to carry out a special selection of the best ones in order to form 'marrow' (73B). Thus once the creation is over, the elementary triangles are fixed. If even the Demiurge has to select, rather than remaking or smoothing off, no later process is going to disturb them. However, Visintainer's continuous division demands a very severe disturbance indeed.

It is claimed that this work is following an interpretation by Bruins (1951); specifically, in the above dissection of elementary triangles it is stated that (Visintainer 1998, p. 122):

> Then, following the scheme of Bruins (1951), we could see how, for any given equilateral triangle, it could become itself a tetrahedron, given that it should 'cut itself' in the correct way and 'fold out' into the third dimension (Fig. 10).

This misrepresents Bruins in several ways. Nowhere does he or Plato mention any unit 'cutting itself'. What is discussed at this point is the exact inverse of what is claimed, an *un*folding (*rabattement/développment*) of a tetrahedron, which has been cut *along edges*, into a four-fold equilateral triangle. The important point here is that Bruins everywhere assumes that his equilateral triangles or squares are made up of smaller units, scalene or isosceles, which are *a-tomon*, and in this he is in agreement with the above quotation from Taylor (1928). It is inconceivable that he would countenance a folding *across* the elementary units as is shown by Visintainer in his Figures 9 and 10. Later, where Bruins demonstrates a division of a cube, he refers to the cutting of the other polyhedra "*along the edges of the right-angled triangles*",[9] again echoing Taylor. For the cube, he makes it equally clear in his diagram that the cutting is happening along the edges of pre-existing isosceles triangles, and he uses this as a reason to argue in favor of the *Timaeus* construction of the square. Some of his planar units do indeed fold up during interconversions, but only along pre-existing join lines. It is very clear that his implicit model for the way in which larger units have been constructed follows, in principle if not in detail, that of Cornford, the very model which Visintainer claims is being overturned. For Bruins, in dividing down or in folding, there will be a clear limit when the edges of the elemental triangles are reached. The procedure described, which cuts or folds across these, is in no way 'following' Bruins.

A related point against the idea of continued division is that there are serious questions about how it could work. It seems to need a supernatural agency, such as the Demiurge continually interfering in his creation, and the evidence in the text is in the opposite direction: after the creation, he withdraws from it ("he proceeded to abide at rest in his own customary nature", 42E). Without this, it is difficult to see a mechanism for the process of "cutting", which will need to be very precise if it is to produce "matched sets" of smaller triangles suitable to build Plato's regular solids. Since Plato speaks of "battles" between particles of his elements (81C-D, 56B), and is clearly thinking of violent impacts, presumably random, we might expect random fragmentation rather than precise cutting. If there are pre-existing cleavage lines, then there must be smaller finite sub-units present, beyond which no further cutting can take place.

Finally, there is an argument from Cornford which is relevant. At 57C9 the phrase "*tôn stoicheiôn aitiateon sustasin*" occurs. Although *sustasis* can have the passive meaning of structure, Cornford comments that an active meaning, of putting together, is the only possible meaning for the text. His translation reads "The reason why there are several varieties within their kinds lies in the construction of each of the two elements". The 'kinds' refer back to the 'uncompounded and primary bodies'. Thus different sizes arise from different assemblies/constructions; this is alluded to briefly in the second part of this paper. However, in Visintainer's scheme involving continuous divisibility, there would be no reason to have such a difference of assembly. The process of 'putting together' would be the same for any size of triangle, so that this part of the *Timaeus* text would be at least redundant, if not meaningless. Thus Cornford's 'building block' approach can still stand, and I now turn to an analysis of one aspect of this.

## Part II: Symmetry in the Assembly of the Polyhedra

### 1. The problem at *Timaeus* 55D-E

As noted in the introduction, the text of the *Timaeus* at this point specifies sub-units, squares, and equilateral triangles, which are more complicated than might be expected. A square can be built from two isosceles right-angled triangles, but four are specified; an equilateral triangle can be constructed from two (30°, 60°, 90°) scalene triangles, but six are specified.

Cornford (1937) suggested that Plato's intention was that many different constructions are to be understood for the square and equilateral, all based on different numbers of the original triangles, and that the text gives us only a 'sample' of each. This proposal arises from another feature of Plato's elements, that they exist in various sizes with different properties (57C-D, 58C-D). Cornford went further by proposing that in each case Plato's sample is the *second* member of a series which begins with the simpler form. It is this second part of his proposal which presents problems, since it is odd that Plato should give us no indication in his text that this is his intention. In the subsequent literature the usual view has been that of Zeyl (2000, p. lxviii), that there is still "something of a mystery " as to why Plato chose the more complex scheme. Using intuitive arguments, I have shown elsewhere (Lloyd 2006) that there is no mystery; all the triangles specified are required, so long as the symmetry of the final body to be constructed is considered at earlier stages. This article includes a fuller survey of the literature than is necessary here, and references to occasional suggestions by others that symmetry might be significant for this problem; Rex (1989) also indicates the importance of rotational symmetry.

An intuitive approach, based on visualization of these polyhedra ('bodies', *sômata*, in *Timaeus*), possibly including the examination of models, may well have been what prompted the original constructions in the *Timaeus*. Here I show that the same constructions also follow from an elementary application of Group Theory, at the level of many undergraduate chemistry courses. I do not intend to suggest that symmetry in its modern sense is implied in the *Timaeus*; the word *symmetria* has much more the sense of our 'proportion' than of 'symmetry'. Here I intend merely to demonstrate that, whatever the original reasons for the choice, it was the correct one.[10] Only by following Plato's specification can polyhedra with the true symmetry of the regular bodies be constructed. If the *Timaeus* specifications are not followed, many isomeric forms result for each size of each element, and some analysis of this problem is also presented here. If they *are* followed, there is only a single 'isomer', a unique form.

## 2. Symmetry analysis

The majority of modern undergraduate textbooks of Inorganic and Physical Chemistry include a summary of the necessary group theoretical language. I give a very brief introduction here, generally following the approach of Shriver and Atkins 1999, pp. 117-8. This reference may be consulted for further explanation, and more detail is available in Harris and Bertolucci 1978, and in Cotton 1990. The essence of the approach is to examine the complete set of symmetry *operations* associated with some particular physical unit, often a molecule, but here I apply it to four of the Platonic Solids and their constituent triangles. This set of operations forms a group, which specifies fully the symmetry of the unit. Knowledge of the mathematical properties of any group, particularly as summarized in "Character Tables" (which give traces of transformation matrices), allows wide-ranging rigorous conclusions to be drawn about the physical properties of the unit concerned. The operations transform the unit into exactly equivalent configurations; they are associated with particular symmetry *elements*, such as an axis of rotation, but the set of *operations* constitutes the group.

The dimensionality of the triangles used by Plato is not immediately clear. A few commentators assume them to be at least implicitly three-dimensional, but most have followed Aristotle, accepting that they are true plane figures. The symmetry analysis is simplified by ignoring any operations which involve the third dimension; in any event the additional symmetries which would result from including these are destroyed in the later assembly of the polyhedra, since the inner and outer sides of the triangles then differ.

### *(a) The equilateral triangle*

The Cornford construction of an equilateral triangle by assembling two scalene triangles is shown in Figure 1a, and that defined by Plato (*Timaeus*, 55D-E) is shown in Figure 1b. Figure 1c shows an undivided triangle; this is converted to an equivalent configuration by rotation about an axis, perpendicular to the paper, through 120°. This can be repeated, and a third repetition regenerates the original configuration. The axis is called 'three-fold', and is labeled as $C_3$. If there are no other operations in the group, the same label is also used for the group; specifications of labels for more complex cases can be found in the above references.

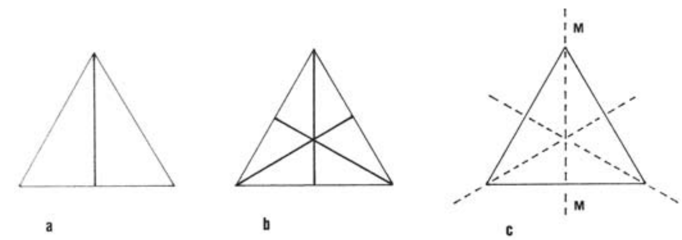


Figure 1. (a) Construction of the simplest possible equilateral triangle, according to Cornford; (b) construction according to *Timaeus*, 55D-E; (c) The mirror planes of an equilateral triangle.

In addition to these rotation operations, such a triangle has left-right symmetries. The corresponding operation is that of an imaginary mirror plane, again perpendicular to the triangle. The mirror at MM in Figure 1c transforms all points in the right-hand half into corresponding points in the left-hand half, and vice versa, and there are two other such planes through the other vertices. All three intersect at the same point, which is also the rotation center. The set of all rotation and reflection operations specifies the group, one of the 'point groups'; the appropriate label is $C_{3v}$.[11] Further details of this and other point groups are given in the references already cited.

We can also apply the operations of $C_{3v}$ to the composite triangle of Figure 1b. We consider the join lines, the only new feature of Figure 1b, as an 'object'. However, this 'object' has exactly the same symmetry elements, and operations, as the equilateral triangle, so it also has $C_{3v}$ symmetry. Thus the mirror plane MM (see Figure 1c) converts the vertical join into itself, and interchanges the other two. The other two mirror planes behave similarly, and the 'object' also has three-fold rotational symmetry. Thus the complete composite triangle behaves in exactly the same way as an undivided equilateral triangle under the operations of the $C_{3v}$ point group. When the composite triangle was assembled, it acquired the full symmetry of an undivided triangle, and Plato's construction, Figure 1b, is completely equivalent to an undivided equilateral triangle in symmetry terms. I have argued elsewhere (Lloyd 2006) that Plato's construction, 55 D-E, is equivalent to taking one of the elemental triangles and successively operating on it, first with a mirror plane, then with the 3-fold axis.

In stark contrast, Cornford's triangle 1a has a much lower symmetry – only one reflection plane remains, and the point group is $C_s$.[12]

### *(b) The tetrahedron*

Assembly of a set of four equilateral triangles (*Timaeus,* 54e-55a) generates the tetrahedron, with point group $T_d$. The symmetry elements of the original triangles are now present in, and operate on, the entire tetrahedron surface. Plato is only concerned with faces, and not with the interior. If triangles according to Figure 1b are used for this assembly, then by the same argument as was used above for the equilateral triangle, it follows that the point group of the set of join lines over all four faces is exactly the same as that of a tetrahedron with undivided faces, so that the $T_d$ symmetry is unchanged by the presence of the join lines.

Some of the effects of the operations of the symmetry elements can be seen in Figure 2. This shows a tetrahedron, made up from Plato's triangles (according to Figure 1b), in projection down along one of the four 3-fold axes; faces 1-3 are facing the viewer. The join lines on face 4, the base, are obscured by the three faces above, but the operations of this 3-fold axis carry out the same transformations as before on the base triangle.

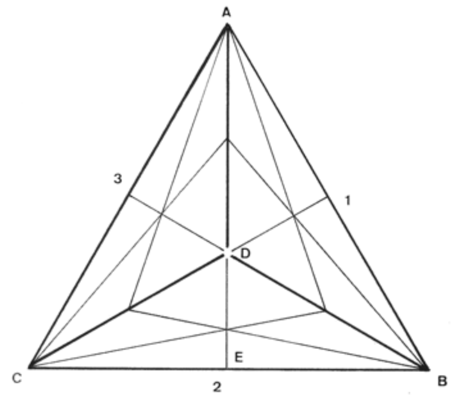


Figure 2. The tetrahedron ABCD shown in projection, perpendicular to the 3-fold axis which runs through D. Each side is an equilateral triangle constructed as in Figure 1b.

A single rotation operation also transforms face 1 with its set of join lines into face 2, *etc.* The mirror plane passing through edges AD and DE transforms 1 into 3, and exchanges the left and right halves of 2.

The operations of the elements of the individual faces on each other generate new symmetry elements of the three-dimensional point group $T_d$, the $S_4$ axes. These, which include two-fold axes, run through the centers of opposite edges of the tetrahedron (and therefore of joins at triangle edges according to Figure 1b), so their operations also transform the set of joins into itself. In order to see this, it is useful to imagine the tetrahedron inscribed in a cube (Figure 3a). The vertical $S_4$ axis transforms edge AB into DC, and face 2, with the set of join lines, into face 4, *etc.* Similar comments apply to the operations of the other two perpendicular $S_4$ axes.

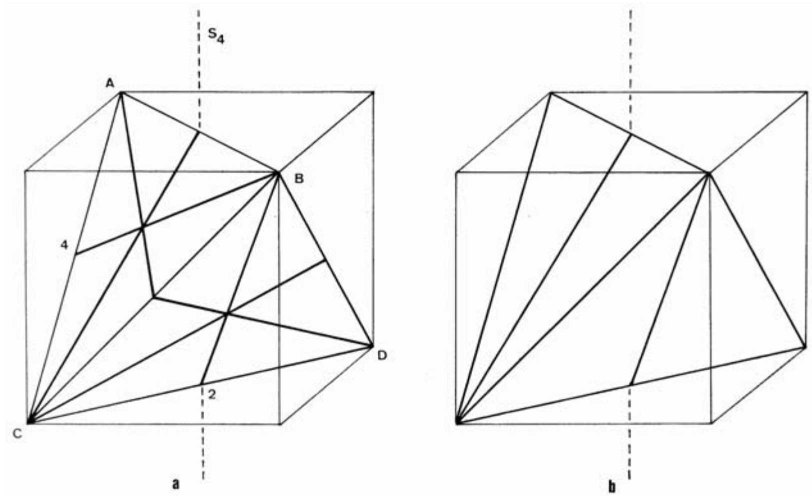


Figure 3. The operation of $S_4$ axes on tetrahedra: (a) constructed from triangles according to Figure 1b; (b) constructed from triangles according to Figure 1a.

However, any attempt to construct tetrahedra using triangles according to Figure 1a fails to generate $T_d$ symmetry. The best that can be obtained is $D_{2d}$, shown in Figure 3b. One $S_4$ axis is retained, and this interconverts the four join lines, only two of which are shown, into each other. However, the other $S_4$ axes, and many of the other symmetry elements of the regular tetrahedron (particularly those of the three-fold axes) have been lost. In addition, so has the uniqueness of Figure 3a. All possible orientations of all the triangular faces still give the same regular tetrahedron for Figure 3a, but this is not so for Figure 3b, which generates a variety of 'isomeric' forms.

### *(c) Isomers*

There are very many forms of these low-symmetry tetrahedra, and the modern molecular concept of isomers is appropriate to describe them. Each triangle according to Figure 1a can be set independently in any of three orientations, so there is a total of 81 possible orientations. There are equivalent sets of these, which cover several of the sub-groups of $D_{2d}$. These sets can be considered to arise from the equivalence of the regular tetrahedron with respect to the x,y,z axes of Cartesian geometry; the two-fold axes are conventionally aligned with these (in Figure 3 the z axis would normally be vertical, so that it includes the $S_4$ axis shown). The general case is of an arrangement of joins which has no symmetry, point group $C_1$. For any pattern of these joins seen from the +x direction, there is an exactly equivalent pattern as seen from -x, which arises from a different arrangement of join lines on the tetrahedron. The same is true for y and z, so there is a set of six equivalent structures; this is also true for $C_s$, $C_2$ and $S_4$ symmetries. If a structure does not have a reflection-rotation axis (Atkins 1986, p. 409), as is true for $C_1$, $C_s$ and $C_2$ symmetries, then there are also optical isomers, so that there is a total of twelve for each set which has one of these symmetries. However, in the case of the set of $S_4$ type, there are only six, since this cannot have optical isomers. For the $D_{2d}$ type oriented along ±z, there is only one structure, since the pattern as seen from +z is identical with that from –z, and so there are only three tetrahedra with this symmetry. By inspection[13] the total of 81 is made up of six sets of twelve equivalent tetrahedra (four of type $C_1$, one each of $C_s$ and $C_2$), one of six ($S_4$), and one of three ($D_{2d}$), giving a total of eight distinguishable isomeric forms. From the *Timaeus* text, it seems most unlikely that these multiple isomeric forms would have been acceptable to Plato (Lloyd 2006).

### *(d) The octahedron and icosahedron*

The octahedron constructed with triangles according to Figure 1b has the full $O_h$ symmetry of an undivided octahedron. The three-fold axes of the triangles become the four three-fold axes of the $O_h$ point group, running through opposite pairs of faces. Again, these rotate the individual join lines in these faces into one another, and rotate the other six faces, with their join lines, into each other. The mirror planes of the triangles remain, and new ones are created in the assembly of the octahedron, which run through edges. In projection onto a plane perpendicular to a four-fold axis, the outline becomes a square. Each operation of the four-fold axis rotates each triangle, with its join lines, into the next one (compare the operation of the 3-fold axis in Figure 2; an illustration of the operation of the 4-fold axis is given in Lloyd 2006 as Figure 9). The mirror planes coincident with edges also interchange all triangles, whereas those which divide faces carry out the same operations on these as they do on the constituent triangles, and interchange the other faces. The analysis can be continued to all the operations; in every case the join lines are converted to equivalent configurations.

However, if triangles according to Figure 1a are assembled, the highest possible symmetry, with all the join lines oriented towards one of the four-fold axes, is $D_{4h}$. A structure of this type has appeared in the discussion of the Cornford-derived polyhedra shown in Friedlander 1958 and also in Vlastos 1975 (pp. 69-79), but apparently without the low symmetry having been noticed. Possible reasons for the loss of symmetry not being detected earlier are discussed in Lloyd 2006. There are many more isomers than for the tetrahedron.

Similarly, a unique, completely regular icosahedron, with point group $I_h$, can be assembled from triangles according to Figure 1b. However, the maximum symmetry accessible if triangles according to Figure 1a are used is $D_{5d}$. This low-symmetry object has been illustrated in Vlastos 1975 (Figure 4, p. 76); again there are very many isomers.

### *(e) The square and cube*

A true square has $C_{4v}$ symmetry, if we restrict the operations to two dimensions, and this is also true of Plato's specification in *Timaeus* (55B-C) as one made up from four right-angled isosceles triangles (Figure 4b). Cornford's 'simpler' square (Figure 4a) has a symmetry reduced to $C_{2v}$. The point group of the regular cube is $O_h$, and this follows automatically for a cube constructed from the four-component ($C_{4v}$) square. However if the two-component square is used, several symmetries are possible, of which the highest is $D_{3d}$. This can be obtained in two different arrangements, one of which is probably implied in the left-hand item in Figure 6 of Vlastos 1975 (p. 78). Lower symmetries produce many different isomers, so the square in Figure 4a is no more suitable for building a unique symmetric cube than is the triangle in Figure 1a for building fully symmetric deltahedra.

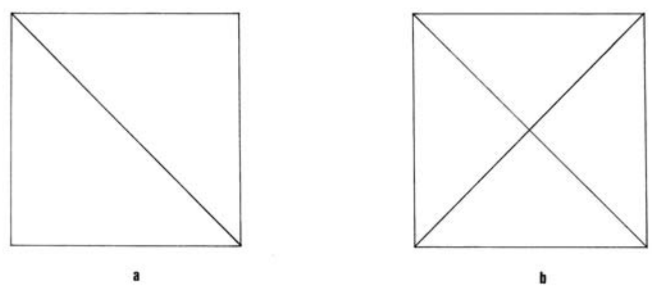


Figure 4. Construction of the simplest square from isosceles right triangles: (a) according to Cornford; (b) according to *Timaeus*.

## 3. An alternative approach

Rather than first constructing the equilateral triangles (squares) of the faces, it may be appropriate to begin with the Platonic solids, with each face divided as above. Professor P. W. Fowler has pointed out that the full Platonic schemes for the tetrahedron, octahedron, and icosahedron, using triangles according to Figure 1b as faces, in each case generate the regular representation of the appropriate group. (A brief description of the language used here is given by Fowler and Quinn (1986)). The individual scalene triangles are irreducible regions, transformed into each other (permuted) by the operations of the particular group; any point within one of these triangles, when operated on by the group, generates the regular orbit. The regular representation has a character under the identity equal to the order of the group, so that the number of elementary triangles is equal to the number of operations, *i.e.* 24 for the tetrahedron, 48 for the octahedron, and 120 (as is specified by Plato at 55A-B) for the icosahedron. This provides a very economical demonstration of the point established rather more laboriously above, that Plato's method for constructing the faces of these deltahedra generates bodies with the full symmetry of the regular, undivided bodies.

The argument is not exactly analogous for the square used at 55B, since the isosceles triangles used in Figure 4b are not irreducible units; to generate the regular representation, all four of these triangles have to be divided along the bisecting mirror planes of the square. Plato's construction here uses pairs of these irreducible units. It would probably have seemed unnecessary to him to make this final division, since the square according to Figure 4b is adequate for his declared purpose of using an isosceles triangle (54A-B) as the basic unit.

## 4. Larger species

It was noted in the Introduction that Plato requires various sizes of each of his elements; this was Cornford's original motivation for his set of constructions. The first of these, as in Figure 1a (or 4a) has too low a symmetry, or, equivalently, it does not follow Plato's construction methods at 54D-E and 55B-C. Nevertheless, Cornford's argument that Plato intended his readers to understand that the different sizes of triangles were all made up from combinations of the same basic units is unaffected by this. The only stipulation must be that such larger units can create polyhedra which have the full symmetry of the appropriate point group. I have shown elsewhere (Lloyd 2006) that there are composite triangles, of increasing size, which satisfy this condition. That there must be some larger suitable triangles is most readily demonstrated, in the context of Section 3 above, by the observation that any individual scalene triangle in one of the deltahedra is an irreducible region of the polyhedron. This is equally true if each scalene triangle is a composite, built up from smaller units. Any such composite scalene triangle, with angles of 30°, 60°, and 90°, can be used as the irreducible unit. The operations of the point group on this will generate first a large equilateral triangle of the type of Figure 1b, and then the regular polyhedron. The first of these composite scalene triangles has three components (Figure 1b can be thought of as made up of two of these, divided by the plane MM), and the next has four. The figures in Cornford 1937 on pp. 236-7 develop some equilateral triangles which are made up in this manner, and these inevitably have the correct $C_{3v}$ symmetry. However his method of constructing equilateral triangles also generates many which cannot be decomposed into a set of six equivalent scalene triangles, and have a lower symmetry. In contrast, Plato's construction at 55D-E, summarized in Figure 1b, can be used to generate any desired size of equilateral triangle by choosing an appropriate composite scalene triangle: all equilateral triangles developed by this method have to be suitable for constructing truly regular deltahedra. Similar comments apply to larger squares. With this set of larger triangles and squares, the many different elements which are required by Plato's scheme for chemistry can be constructed.

## Conclusions

Visintainer's suggestion of dividing the elemental triangles is at variance with the text of the *Timaeus*, with the evidence of Aristotle, and with a variety of comments in the literature. His argument for a continued division, based on *Timaeus* 57D, is not sustainable; this text cannot carry the meaning attributed to it. There is no reason to suppose that Plato would have thought it in any way necessary, desirable, or even possible to have a division of one of his elementary triangles. The proposal, or 'assertion', ascribed to Cornford, that there are two, and only two, minimally sized triangles in Plato's scheme, of finite size, can stand. The attack on Cornford's approach would, if successful, have demolished his particularly significant 'building block' idea, of constructing larger triangles by assembling various numbers of the minimal *stoicheia*; this idea is unaffected.

However, in putting forward his particular scheme, Cornford (1939, p. 237) poses the question: "Why, then, did Plato analyze his equilateral face into six elements, which have less *prima facie* claim to perfection [than a simple equilateral triangle]?" The result of the symmetry analyses above is that the six-fold unit has precisely the 'perfection' of the simple equilateral triangle, and is therefore appropriate to construct truly regular polyhedral elements, of symmetry $T_d$, $O_h$ or $I_h$. This is *not* true of Cornford's proposed two-fold unit, the triangle in Figure 1a. Use of this smaller unit leads both to asymmetry and to multiplicity, to 'isomers', for all three deltahedra. Exactly parallel comments apply to the squares and cubes generated from them. The argument can also be extended to include larger polyhedra, so long as Plato's constructions are adhered to. It is implicit in this that there is only a need for one size of each of the two types of elementary triangles. Cornford's principle of using 'building blocks' to assemble the faces of larger polyhedra, which form the various sorts of each type of element, gains further support from this.

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## Notes

1 Throughout this paper I use the translation by Zeyl (2000), unless otherwise specified, and I refer to the accompanying commentary, as well as to those of Cornford (1937) and Taylor (1928).

2 The reception of the text in antiquity is summarized by Zekl (1992, pp. lxxi-lxxiii). Zeyl (2000), gives a historical perspective leading up to the present day, see particularly pp. xiii-xv.

3 Plato would not consider the term 'element' appropriate for describing his bodies (see *Timaeus* 48B-C) because he does not believe that these are fundamental, but the convention is so well established that I will use it here.

4 Possibly in other languages also. Zekl (1992, p. xiv) states that his extended essay and extensive notes to his translation do not constitute a commentary to replace Cornford 1937 and earlier commentaries.

5 See in particular Zeyl 2000, where there is no indication that there are alternative views. For alternatives, see Mugler 1960, pp. 21-26; Böhme 2000, p. 308; Brisson 2001, pp. 303-6. These will be discussed elsewhere.

6 There is of course an earlier mention of division of a surface into triangles, and dividing these into right-angled triangles, at 53C-D. However, this is establishing a general point about the nature of plane surfaces, not an invitation to divide these right triangles further.

7 Mugler (1960) has made a similar claim to Visintainer about the interpretation of the Greek text, but this has been strongly criticized by O'Brien (1984).

8 Here Cornford 1937 has the disconcerting translation "the number of these differences being the same as that of the varieties in the kinds". If taken literally, this is mathematical nonsense, but the word 'difference' is unnecessary. Archer-Hind 1888 has "and there are just so many sizes as there are kinds in the classes", Bury 1966 "of sizes as numerous as are the classes within the Kinds", and Jowett 1892

"there are as many sizes as there are species of the four elements". Cornford's 'difference' can only be interpreted as meaning 'different sort'.

[9] My italics; the original reads "suivant les côtés des triangles rectangles" (Bruins 1951, p. 279, '*Remarque*').

[10] Nevertheless, it may be noted that Brisson (2003) has suggested that Plato may have had some idea of our axis of symmetry; he writes (p.14): "on peut cependant penser que […] Platon veut trouver un centre de symétrie axiale". I have argued elsewhere (Lloyd 2006) that passages from Plato's nephew Speusippus suggest some sort of awareness of our 'symmetry' concept.

[11] If the additional operations in three dimensions are considered, this becomes $D_{3h}$.

[12] $C_{2v}$ if the additional operations in the third dimension are considered.

[13] The device of the stereographic projection (Harris and Bertolucci 1978, pp 17ff.) is useful in working through these different structural types.

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*D. Robert Lloyd:*
*Chemistry Department, Trinity College, Dublin 2, Ireland;*
*dlloyd@tcd.ie, boblloyd@waitrose.com*